

Kepler Orbits

Equation of motion for a two-body system

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^2} \mathbf{e}_r = -\frac{\mu}{r^3} \mathbf{r} \quad (1)$$

with the gravitational parameter $\mu \equiv G(m_1 + m_2)$ and $\mathbf{r} = \mathbf{r}_{12}$ position of m_2 with respect to m_1

Specific angular momentum is constant

$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}} = rv_{\perp} \mathbf{e}_h \quad (2)$$

with velocity $\dot{\mathbf{r}} = \mathbf{v} = v_r \mathbf{e}_r + v_{\perp} \mathbf{e}_{\perp}$ and $v = |\mathbf{v}|$

Eccentricity vector is constant

$$\mathbf{e} = \frac{\dot{\mathbf{r}} \times \mathbf{h}}{\mu} - \frac{\mathbf{r}}{r} = \left(\frac{v^2}{\mu} - \frac{1}{r} \right) \mathbf{r} - \frac{\mathbf{r} \cdot \dot{\mathbf{r}}}{\mu} \dot{\mathbf{r}} \quad (3)$$

with the eccentricity $e = |\mathbf{e}|$ of the conic section

$e = 0$	circle	$a = r > 0$
$e \in (0, 1)$	ellipse	$a > 0$
$e = 1$	parabola	$a \rightarrow \infty$
$e > 1$	hyperbola	$a < 0$

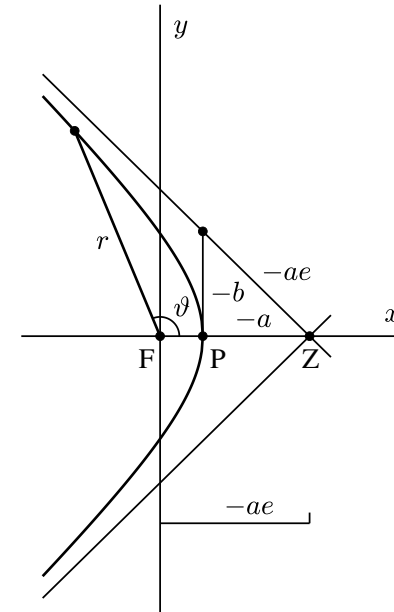
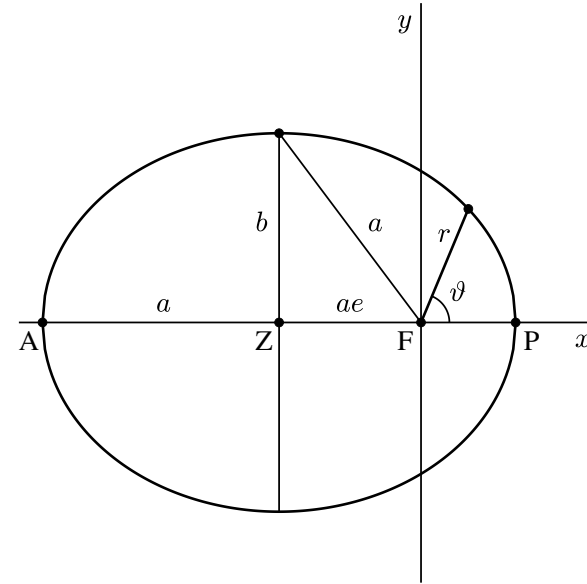
Orbit equation

$$r = \frac{h^2}{\mu} \frac{1}{1 + e \cos \vartheta} = \frac{p}{1 + e \cos \vartheta} \quad (4)$$

where ϑ is the true anomaly,

$p \equiv h^2/\mu$ is the semilatus rectum and

a is the semi-major, b the semi-minor axis of the conic



Specific energy is constant

$$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2p} (1 - e^2) = -\frac{\mu}{2a} \quad (5)$$

with velocity $v = |\mathbf{v}|$ along the orbit

Vis-viva equation

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a} \quad (6)$$

Radial and perpendicular velocity components

$$v_r = \frac{h}{p} e \sin \vartheta \quad (7)$$

$$v_{\perp} = \frac{h}{p} (1 + e \cos \vartheta) \quad (8)$$

Kepler's third law of planetary motion

$$T^2 = \frac{4\pi^2}{\mu} a^3 \quad (9)$$

gives the orbital period T for a body in a closed orbit

Position as a function of time in a circular orbit

$$\vartheta - \vartheta_0 = n (t - t_0) = \sqrt{\frac{\mu}{r^3}} (t - t_0) \quad (10)$$

with the constant angular velocity $n = v/r$ and the initial condition $\vartheta_0 = \vartheta(t_0)$

Position as a function of time in an elliptic orbit

Relationships between true, excentric and mean anomaly and time

$$\tan \frac{\vartheta}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \quad (11)$$

$$E - e \sin E = M \quad (12)$$

$$M \equiv \frac{2\pi}{T} (t - t_0) \quad (13)$$

with the mean anomaly at periapsis passage $M_0 = M(t_0) = 0$

Orbital equation in terms of eccentric anomaly

$$r = a(1 - e \cos E) \quad (14)$$

Position as a function of time in a hyperbolic orbit

Relationships between true, excentric and mean anomaly and time

$$\tan \frac{\vartheta}{2} = \sqrt{\frac{e+1}{e-1}} \tanh \frac{E}{2} \quad (15)$$

$$e \sinh E - E = M \quad (16)$$

$$M \equiv \frac{h}{ab} (t - t_0) \quad (17)$$

with the mean anomaly at periapsis passage $M_0 = M(t_0) = 0$

Orbital equation in terms of eccentric anomaly

$$r = a(1 - e \cosh E) \quad (18)$$

Orbits in Space – Euler Angles

Direction of the ascending node

$$\mathbf{k} = \mathbf{e}_z \times \mathbf{h} = \begin{pmatrix} -h_y \\ h_x \\ 0 \end{pmatrix} \quad (19)$$

Longitude of the ascending node

$$\cos \Omega = \frac{\mathbf{e}_x \cdot \mathbf{k}}{k} = \frac{-h_y}{\sqrt{h_x^2 + h_y^2}} \quad (20)$$

if $h_x > 0$, i.e. $k_y > 0$, then $\Omega \in (0, \pi)$,
if $h_x < 0$, i.e. $k_y < 0$, then $\Omega \in (\pi, 2\pi)$

Inclination

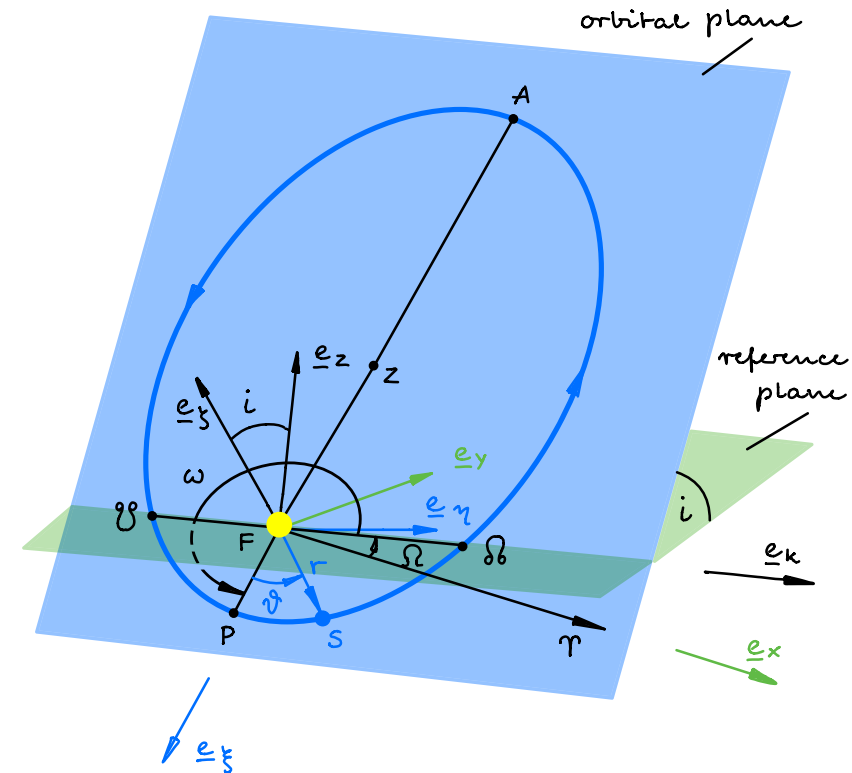
$$\cos i = \frac{\mathbf{e}_z \cdot \mathbf{h}}{h} = \frac{h_z}{h} \quad (21)$$

for $i < \pi/2$ the orbit is prograde,
for $i > \pi/2$ the orbit is retrograde

Argument of the periapsis

$$\cos \omega = \frac{\mathbf{e} \cdot \mathbf{k}}{ek} \quad (22)$$

if $e_z > 0$ then $\omega \in (0, \pi)$,
if $e_z < 0$ then $\omega \in (\pi, 2\pi)$



True anomaly

$$\cos \vartheta = \frac{\mathbf{e} \cdot \mathbf{r}}{er} \quad (23)$$

if $\mathbf{r} \cdot \mathbf{v} > 0$, then $\vartheta \in (0, \pi)$,
if $\mathbf{r} \cdot \mathbf{v} < 0$, then $\vartheta \in (-\pi, 0)$ or $\vartheta \in (\pi, 2\pi)$

Rocket Dynamics

Thrust

$$T = -\dot{m} c \quad (24)$$

with the (negative) mass flux \dot{m} expelled by the rocket and the effective exhaust velocity c

Tsiolkovsky rocket equation

$$v_1 - v_0 = c \ln \frac{m_0}{m_1} \quad \text{or} \quad \frac{m_1}{m_0} = \exp\left(-\frac{\Delta v}{c}\right) \quad (25)$$

gives the velocity change $\Delta v = v_1 - v_0$ achieved by reducing the mass of the rocket from m_0 to m_1 under the assumptions of a one-dimensional motion in force-free space and a constant c

Equations of motion in a central gravitational field

$$\dot{v} = -g \sin \gamma + \frac{T}{m} \cos u - \frac{D}{m} \quad (26)$$

$$v \dot{\gamma} = -\left(g - \frac{v^2}{r}\right) \cos \gamma + \frac{T}{m} \sin u \quad (27)$$

$$\dot{\varphi} = \frac{v}{r} \cos \gamma \quad (28)$$

$$\dot{r} = v \sin \gamma \quad (29)$$

in polar coordinates r, φ with

- v tangential velocity of the rocket
- γ flight path angle
- u pitch angle
- D aerodynamic drag (lift not taken into account)

Total mass of a rocket at takeoff consists of propellant, structural, and payload mass

$$m_0 = m_P + m_S + m_L \quad (30)$$

Mass ratio at burnout

$$Z = \frac{m_0}{m_S + m_L} = \frac{m_0}{m_0 - m_P} \quad (31)$$

defines via eq. (25) the **characteristic velocity** $\Delta v = c \ln Z$

$$\sigma = \frac{m_S}{m_P + m_S} \quad \text{structural coefficient} \quad (32)$$

$$\nu = \frac{m_L}{m_P + m_S} = \frac{m_L}{m_0 - m_L} \quad \text{payload ratio} \quad (33)$$

Optimal staging

A multistage rocket has n stages with the mass $m_i = m_{P_i} + m_{S_i}$ for the i -th stage. If the payload mass m_L and the total characteristic velocity $\Delta v_{\text{tot}} \equiv \sum_{i=1}^n \Delta v_i$ are given (together with c_i and σ_i), the minimum mass $M \equiv \sum_{i=1}^n m_i$ can be calculated by solving the following equations

$$\Delta v_{\text{tot}} - \sum_{i=1}^n c_i \ln \frac{c_i + 1/\lambda}{c_i \sigma_i} = 0 \quad (34)$$

$$Z_i = \frac{c_i + 1/\lambda}{c_i \sigma_i} \quad (35)$$

$$\frac{M + m_L}{m_L} = \prod_{i=1}^n \frac{(1 - \sigma_i) Z_i}{1 - \sigma_i Z_i} \quad (36)$$

Impulsive Thrust Maneuvers

Impulsive orbit transfer

$$\Delta \mathbf{v}_k = \mathbf{v}_k^+ - \mathbf{v}_k^- \quad (37)$$

at the time instant t_k the vehicle undergoes a velocity change from \mathbf{v}_k^- to \mathbf{v}_k^+

Hohmann transfer between two co-planar circular orbits

$$\Delta v_{\text{tot}} = \sqrt{\mu} \left[\left(\sqrt{\frac{2}{r_1} - \frac{2}{r_1 + r_2}} - \sqrt{\frac{1}{r_1}} \right) + \left(\sqrt{\frac{1}{r_2}} - \sqrt{\frac{2}{r_2} - \frac{2}{r_1 + r_2}} \right) \right] \quad (38)$$

where $0 < r_1 < r_2$

Initial phase angle for rendezvous with Hohmann transfer between two co-planar circular orbits

$$\varphi = \pi \left(1 - \left(\frac{1 + r_1/r_2}{2} \right)^{3/2} \right) \quad (39)$$

Synodic period for two objects in co-planar circular orbits

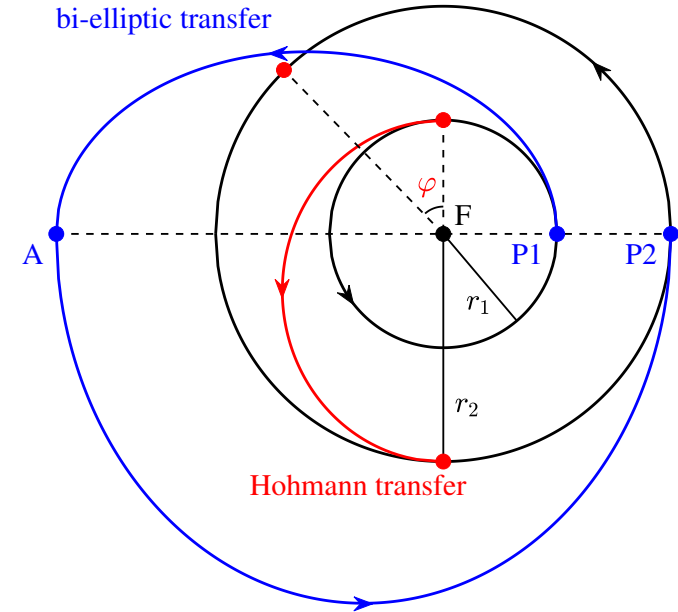
$$S = \frac{2\pi}{n_1 - n_2} = \left(\frac{1}{T_1} - \frac{1}{T_2} \right)^{-1} \quad (40)$$

with the angular velocity $n = v/r = \sqrt{\mu/r^3}$

Bi-elliptic transfer between two co-planar circular orbits

$$\frac{\Delta v_{\text{bi}}}{v_1} = \sqrt{\frac{2(\beta - 1)^2}{\beta(\beta + 1)}} + \sqrt{\frac{2}{\alpha} + \frac{2}{\beta} - \frac{1}{\sqrt{\alpha}}} - 1 \quad (41)$$

where $0 < r_1 < r_2$, $r_2 = \alpha r_1$ and $r_A = \beta r_1$, thus $1 < \alpha < \beta$ and $v_1^2 = \mu/r_1$ is the orbital velocity in orbit 1



One-impulse change of the plane for a circular orbit

$$\frac{\Delta v}{v_1} = 2 \sin \frac{\psi}{2} \quad (42)$$

where $v_1^2 = \mu/r_1$ is the orbital velocity and ψ the angle of the turn

Astronomical Constants

The Sun

mass	$m_S = 1.989 \cdot 10^{30} \text{ kg}$
radius	$R_S = 6.9599 \cdot 10^5 \text{ km}$
gravitational parameter	$\mu_S = G m_S$ $= 1.327 \cdot 10^{11} \text{ km}^3/\text{s}^2$

Elements of the planetary orbits

planet	semi-major axis ¹	eccentricity	sidereal period	inclination ²
Mercury	0.3871	0.2056	87.969 d	$\approx 7^\circ 00'$
Venus	0.7233	0.0068	224.701 d	$3^\circ 24'$
Earth	1.0000	0.0167	365.256 d	$0^\circ 00'$
Mars	1.5237	0.0934	1 y 321.73 d	$1^\circ 51'$
Jupiter	5.2028	0.0483	11 y 314.84 d	$1^\circ 19'$
Saturn	9.5388	0.0560	29 y 167 d	$2^\circ 30'$
Uranus	19.1914	0.0461	84 y 7.4 d	$0^\circ 46'$
Neptune	30.0611	0.0097	164 y 280.3 d	$1^\circ 47'$

¹ semi-major axes are given in au

² inclination with respect to the ecliptic

The Earth

mass	$m_E = 5.974 \cdot 10^{24} \text{ kg}$
radius	$R_E = 6.37812 \cdot 10^3 \text{ km}$
gravitational parameter	$\mu_E = G m_E$ $= 3.986 \cdot 10^5 \text{ km}^3/\text{s}^2$
mean Solar distance	$r_E = 1 \text{ au}$ $= 1.495978 \cdot 10^8 \text{ km}$

Physical characteristics of the planets

planet	radius ³	mass ⁴	sidereal rotation period	axial tilt ⁵
Mercury	0.382	0.0553	58 d 16 h	$\approx 2^\circ$
Venus	0.949	0.8149	243 d (retro)	$177^\circ 18'$
Earth	1.000	1.000	23 h 56 m 04 s	$23^\circ 27'$
Mars	0.532	0.1074	24 h 37 m 23 s	$25^\circ 11'$
Jupiter	11.209	317.938	9 h 50 m	$3^\circ 07'$
Saturn	9.49	95.181	10 h 14 m	$26^\circ 44'$
Uranus	4.007	14.531	17 h 54 m	$97^\circ 52'$
Neptune	3.83	17.135	19 h 12 m	$29^\circ 36'$

³ mean equatorial radii are given in units of R_E

⁴ masses are given in units of m_E

⁵ inclination of the equator with respect to the orbital plane